Compromise Control of Overdetermined Systems

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In the design or analysis of an automatic control system the physical process variables, temperatures, flow rates, and compositions to name but a few, may be separated into three mutually exclusive categories. Thus employing control terminology widely in use today (*1*) the authors shall speak of the uncontrolled variables, the manipulated variable, and the controlled variables. By uncontrolled variables are meant those quantities which cannot be regulated by the controller and whose fluctuations are the primary cause of the disturbances which take place in the system. Similarly the manipulated variable will be that parameter affecting the process which may be directly adjusted by the controller during the course of the operation. And finally controlled variable will apply to an observable which cannot be regulated directly but which, being affected by the other variables, can in principle at least be adjusted indirectly.

In this article the authors consider overdetermined systems, where a single parameter can be used to control several variables simultaneously. As far as is known such a control arrangement has never before been proposed probably because its behavior cannot be measured by the conventional standards of system performance. Yet by the adoption of a probabilistic rather than deterministic point of view it shall be demonstrated that overdetermined control systems are not only entirely feasible but indeed perfectly capable of being evaluated quantitatively and objectively.

CONVENTIONAL CONTROL THEORY: A DETERMINISTIC VIEWPOINT

First of all it is necessary to examine here briefly the conventional approach to control-system analysis in order to see why it has in the past prevented the serious consideration of overdetermined systems. According to the conventional point of view one generally chooses in advance a set point, which is at some fixed value of the variable to be controlled. Then, since any deviations from this set point are thought of as being undesirable, all of the present measures of system performance, settling time, offset, rise time, static error, overshoot, and mean squared error among others, are made to bear a direct relationship to the error, which is simply the deviation, at any time, of the controlled variable from its set value. Af course this error can in principle be held down to zero under certain ideal circumstances no matter what disturbances are upsetting the system. Thus for very slow perturbations this could be accomplished with a feed forward control scheme by measuring the uncontrolled variables and then calculating the exact adjustment of the manipulated variable needed to nullify the effect of the upset. Alternatively any feedback system with an infinite amplification in the controller could theoretically also hold the error at zero in the face of slow changes.

OVER-DETERMINACY: A PROBABILISTIC VIEWPOINT

Nevertheless even under these ideal conditions where the input disturbances are slow compared with the ability of the system to respond to them, it is physically impossible to hold more than one controlled variable exactly at its set point by regulating a single parameter. As a matter of fact a system with more than one controlled variable is, strictly speaking, overdetermined, since at each instant the single manipulated variable is subjected to as many requirements as there are controlled variables. It is probable therefore that what has discouraged any serious study of overdetermined systems in the past is the obvious impossiblity of simultaneously satisfying all these control restrictions.

But one must not exaggerate the importance of eliminating all errors under ideal conditions because it is usually perfectly acceptable to hold each controlled variable not exactly fixed but within some pair of predetermined limits which, to use the terminology of statistical quality control, define the control band. In this way the state of the system at any given instant would be considered satisfactory only if all controlled variables were to be found simultaneously inside their control bands, and, since the uncontrolled perturbations upsetting the system will be taken as random variables, the overall criterion of performance for comparing the various control schemes will be the probability of satisfactory operation. From a probabilistic standpoint then overdetermined systems are not only quite feasible, they are indeed even capable of objective evaluation.

Overdetermined systems occur in practice more often than one might imagine. As a manufacturing plant ages, parts of it sometimes become overdetermined accidentally when new units are attached to the old. Moreover inventory control problems such as the one described in Figure 1 and treated under less general conditions in earlier articles (*2, 3*) arise whenever tank capacities are insufficient to contain uncontrollable fluctuations in supply, demand, and production.

COMPROMISE CONTROL

It is necessary now to consider two existing modes of controlling overdetermined systems and then to propose two novel procedures which usually result in higher probabilities for satisfactory operation. Whereas the conventional approach would consist of holding either the manipulated parameter itself or one of the variables to be controlled exactly constant in the hope that the fluctuations of the remaining controlled variables would not be excessive, the two new techniques, called *compromise control,* attempt to take advantage of all the flexibility in the system by means of a control function which simultaneously relates the manipulated variable to all the measured quantities.

The two compromise control methods to be introduced here differ from each other only in the manner in which the necessary control constants are determined. One scheme involves minimizing a weighted sum of squared deviations of the critical variables, if here the weight for each variable has been set on an intuitive basis inversely proportional to the square of the width of its allowable range. Although the advantage of this scheme is that the control constants for this weighted least squares criterion are easy to calculate, the resulting probability may not be close enough to the maximum

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especially if some of the variables are much more tightly constrained than the others. To handle such cases another compromise method is therefore suggested which involves minimizing the largest of the weighted squared deviations of the critical variables. This minimax method requires the solution of a nonlinear program for which an algorithm has already been proposed (*4, 5*). While it is true that the numerical work is, in this latter instance, much more involved than for the weighted least squares method, the probability of satisfactory operation which is thereby obtained is necessarily higher.

It is realized of course that from a conventional point of view overdeterminacy is to be avoided. Yet the cost of accomplishing this could indeed be considerable, for overdeterminacy can be removed in only two ways: by increasing the number of manipulated variables, as for example by adding heaters or coolers to a temperature control system, or by decreasing the number of critical variables through the use, for example, of enlarged tanks in an overdetermined storage system. Both alternatives would clearly involve a capital expenditure which would be largely unnecessary if the methods proposed in this paper were found effective.

It can be seen therefore that one is sometimes forced to tolerate overdeterminacy simply because the capital needed to correct it is not available. The use of compromise control in such situations will then at least reduce the number and duration of emergencies which are caused by unsatisfactory operation, whereas in production scheduling problems, such as the one in Figure 1, compromise control can increase the length of the planning period during which medium range decisions do not need to be revised in order to avoid short range emergencies. Reference *2* discusses this question in more detail.

The authors do not on the other hand wish to overstate the case for compromise control. Since its dynamic behavior has not yet been studied, it can be recommended only when the external perturbations change slowly in comparison with the ability of the system to respond. It should also be noted that the physical parameters of the system must be known quite accurately because, in the present state of the theory, there is no corrective action (feedback) generated by discrepancies between the actual values of the dependent variables and their desired values. Research is presently being directed toward introducing feedback into overdetermined systems. A final reservation is that while these methods give a probability of satisfactory operation which is usually better than that given by conventional control schmes, they are in no sense absolutely optimal.

The second part of this paper develops in general mathematical terms the theory of overdetermined systems. It then describes the conventional and compromise control schemes and ends with a discussion of the practical problems involved in closed-loop compromise control. The third part gives the results of a study of an overdetermined process, namely a reactor-distillation column system with insufficient material storage capacity. Time plays a special role in this case because of the accumulation of material in the system. This example compares the control constants and the probability of satisfactory operation for the four control schemes which have been presented here. This paper finally discusses the conclusions to be drawn from the numerical study.

THEORY OF OVERDETERMINED SYSTEMS

Definitions

Consider now a system having several exterior variables $x_i (i = 1, 2, \ldots, I)$, which cannot be regulated, and interior variables y_j $(j = 1, 2, \ldots, J)$, which as the x_i fluctuate are to be controlled indirectly by the adjustment of a single manipulated variable m. Thus the dependent variables y_j are functions of the independent variables x_i and of the manipulated variable m. This study will deal with systems which are linearized about some reference state at which the variables x_i, y_j, and m have respectively the values $\bar{x}_i$, $\bar{y}_j$, and $\bar{m}$. Then if Δx_i, Δy_i, and Δm denote the deviations of these variables from the reference state, and if the linear approximation is used

$$\Delta y_j = \sum_i a_{ij} \Delta x_i - c_j \Delta m \qquad (1)$$

where the coefficients a_{ij} and c_j must be obtained from the physical parameters of the system and depend in general on the choice of the reference state.

The external perturbations Δx_j are of course random, uncontrolled variables, and the interior variables Δy_i can therefore only be regulated through the parameter Δm. Ideally Δm would be chosen to make all of the Δy_j vanish, but this is clearly impossible in view of the fact that the system, having J equations with only one unknown, is overdetermined. Now if all of the Δy_j really had to be held at zero. the process would not be controllable unless enough directly controllable variables were added to the system to bring the total up to J, the number of independent equations. But in most practical situations this is hardly necessary, since a limited variation in the dependent variables is generally allowed. So cases will be considered for which the Δy_j are permitted to fluctuate between the nearer limit, and the farther limit. That is

$$|h_j| \leq |h'_j| \qquad (2)$$

with either

$$h_j \leq \Delta y_j \leq h'_j \qquad (3a)$$

or

$$h'_j \leq \Delta y_j \leq h_j \qquad (3b)$$

in accordance with the circumstances.

Then to take advantage of this flexibility Δm will be manipulated as a function of the exterior perturbations Δx_i in such a way as to optimize the performance of the physical process. For simplicity the authors have studied only linear control functions of the form

$$\Delta m = \sum_i b_i \Delta x_i \qquad (4)$$

where the b_i are control constants to be determined. Therefore

$$\Delta y_j = \sum_i (a_{ij} - b_i c_j) \Delta x_j \qquad (5)$$

Probabilistic Measure of Performance

Before the control constants b_i can be determined a measure of performance must be decided upon. The system will be considered to be operating satisfactorily only when all of the fluctuations Δy_j and Δm are within the limits prescribed in Equations (3*a*) or (3*b*). If the uncontrollable perturbations Δx_i are treated as random variables, the dependent variables Δy_j will also fluctuate randomly and a natural criterion of effectiveness for any given set of control constants would be the associated probability of satisfactory operation of the system.

To calculate such a probability one must know the joint probability distribution of all the Δy_j. The assumption that the Δx_i have the normal or Gauss distribution, together with the linearity of the equations (5), will imply that the Δy_j will also be normal. Although it is not essential to do so, it will be assumed for simplicity that all the Δx_i are stochastically independent. One also avoids considering dynamic effects by limiting analysis to those systems which can respond quickly in comparison with the characteristic time scale of the fluctuations Δx_i. The mean value of each variable will be taken as its reference state so that the expected value of each variable will be zero. And finally the supposition is made that the variance σ_i^2 for each of the uncontrolled variables is known.

The variance τ_j^2 of each indirectly controlled variable may now be expressed as a function only of the unknown constants b_i, by averaging the square of Equation (5). It follows that

$$\tau_j^2 = \sum_i (a_{ij} - b_i c_j)^2 \sigma_i^2 \qquad (6)$$

from which, together with the covariances also expressible as functions of the b_i, it is possible to express the probability of satisfactory operation in closed form with the control constants b_i as the only unknown parameters. One could in principle obtain the control constants maximizing this probability by differential calculus, because the function is continuous and differentiable; but the expression, a definite multiple integral with the parameters b_i in a quadratic exponent, is so complicated that this direct approach is impractical.

Although it would naturally be best to use the control constants which yield the highest possible probability of satisfactory operation, any set of b_i giving a probability close to unity would be acceptable. The authors will therefore discuss several nonoptimal control schemes, each of which gives by a straightforward calculation a set of control constants b_i, and then they will test the effectiveness of any set of b_i by determining the variances τ_j^2 from Equation (6) and, from a table of the normal distribution, the probability that each Δy_j would lie inside its limits. The least of these probabilities will naturally be an upper bound on the probability of satisfactory operation for the process, and since the Δy_j will tend to be highly correlated by the common action of the manipulated variable Δm, one would expect that the system probability would usually be close to this upper bound. A lower bound, which would be exact only if all the Δy_j were stochastically independent, is given by the product of all the individual probabilities. Thus if P_j is the probability that Δy_j will be within its limits, the probability P of satisfactory operation is bounded by

$$\prod_j P_j \leq P \leq \min_j (P_j) \qquad (7)$$

The usefulness of Equation (7) naturally lies in the fact that the bounds are much more easy to calculate than P itself.

Four control schemes will now be presented in ascending order of complexity. It is recommended that each scheme be tried in turn until either the probability is high enough or until all four methods have been investigated.

Control of the Manipulated Variable

The simplest control strategy, which would probably be rather ineffective in general, would be to hold Δm at zero. In this case

$$b_i = 0 \qquad (8)$$

and the variances τ_j^2 of the indirectly controlled variables y_i are obtained by combining Equations (6) and (8) to yield

$$\tau_j^2 = \sum_i a_{ij}^2 \sigma_i^2 \qquad (9)$$

Control of a Single Dependent Variable

On the other hand one of the dependent variables, say y_k, can always be held constant by properly choosing the control parameters. Thus if

$$b_i = a_{ik}/c_k \qquad (10)$$

Equation (6) reduces to

$$\tau_j^2 = \sum_i (a_{ij} - a_{ik} c_j/c_k)^2 \sigma_i^2$$

so that, when $j = k$

$$\tau_k^2 = 0 \qquad (12)$$

where the variable y_k should naturally be one of the more tightly constrained ones. This control method is, because of its simplicity, rather widely used in practice and would be preferable to the more elaborate techniques to be discussed below if it were to result in a high enough probability of satisfactory operation.

A selective or gated control system is used to regulate some overdetermined systems in practice. In such a control system, a particular dependent variable is chosen to be regulated under normal conditions. Whenever some other variable starts leaving its control band, the regulator is switched so it controls this new variable and relinquishes control of the original one. Since at any particular moment only one variable is under command of the regulator, a gated system may be considered a modification of the idea of holding one variable constant.

Such a system has been proposed by Mamzic (6) for controlling the discharge pressure of a pump. In normal operation a control valve downstream from the pump is adjusted to regulate the discharge pressure. Whenever the suction pressure drops so low that it endangers the pump, the regulator stops controlling discharge pressure and reduces the flow until the suction pressure is back within safe operating limits, at which time the regulator again starts to control discharge pressure.

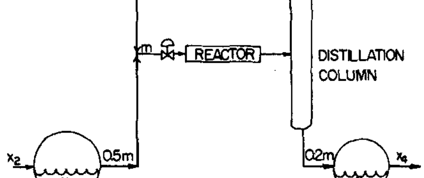

Fig. 1. An overdetermined production control system.

In this application the gated regulator is more of a safety device than the sort of compromise controller defined in this paper, since the operation would not be considered satisfactory during emergencies owing to the necessity of closing the control valve and reducing the pressure delivered.

It is conceivable that an overdetermined system might be controlled successfully by a gated regulator if there were not very many dependent variables to be held within limits. But the sudden switching of control from one variable to another could conceivably upset the system unduly. Since analysis of gated systems would appear to involve rather complicated probabilistic reasoning, at present it is not possible to compare the performance of gated systems with that of the compromise systems to follow.

Weighted Least Squares Control

This is a novel method which attempts to take advantage of the flexibility of the system and the fact that the Δy_j may be allowed to fluctuate between fixed limits. The main idea behind this approach is that, since the normal distribution is such that the probability of Δy_j being within its limits increases as τ_j/h_j, it should be reasonable to require that each quantity τ_j^2/h_j^2 be made as small as possible. Naturally it is impossible to choose the constants b_i in such a way as to minimize all of these quantities simultaneously, but it is possible to minimize their sum:

$$S = \sum_j \tau_j^2/h_j^2 = \sum_j \sum_i (a_{ij} - b_i c_j)^2 \sigma_i^2/h_j^2 \qquad (13)$$

This practical rather than rigorous approach will be called the *method of weighted least squares,* since each variance τ_j^2 has been weighted by the squared reciprocal of h_j. The advantage of this method over the minimax technique to be described shortly is that the minimizing constants b_i can be found in closed form as functions of the known constants simply by setting all the partial derivatives $\partial S/\partial b_i$ equal to zero. It may easily be shown that

$$b_i = (\sum_j a_{ij} c_j/h_j^2) / (\sum_j c_j^2/h_j^2) \qquad (14)$$

Often this control scheme results in a probability of satisfactory operation much higher than that of the two preceding methods, although at the expense of a somewhat more complicated control system. The details of the control setup will be examined after a discussion of a fourth control scheme.

Minimax Control

If the individual probabilities P_j of each y_j lying within its limits are first calculated for one of the preceding control systems, generally one of these probabilities, say P_1, will be lower than all the rest. This smallest probability will usually limit the overall performance of the system. Now P_1 can always be increased by altering the constants b_i so as to decrease the quantities $(a_{i1} - b_i c_i)^2$ in Equation (6) with $j = 1$. Naturally such a change might decrease some of the other probabilities, but if the adjustment is not too great one would expect an improved overall probability of satisfactory operation. Thus any control scheme increasing the smallest probability P_1 would, except under unusual circumstances, also be superior to any of the three preceding techniques.

One should now make the largest of the quantities τ_j/h_j as small as possible since it is rather cumbersome to work directly with probabilities. This approach will be called the *minimax control scheme* because one seeks the constants b_i minimizing the function

$$\max_j \{\tau_j^2/h_j^2\}$$

It should be noted that the above is a piece-by-piece quadratic function of the b_i because it is a multifaceted surface formed by the intersection of the J paraboloids satisfying Equation (6). Since this expression is not differentiable at the intersections, which in general include the minimum itself, it cannot be minimized by the usual rules of differential calculus.

This minimization is in fact a problem in nonlinear programing which may be solved by a straightforward if somewhat tedious algorithm (*4*). It is not practical to make the computations manually for problems having more than three uncontrolled variables x_i, but a recently developed IBM-650 computer program will solve four variable problems in about 5 min. (*5*). One interesting property of the minimax method is that there will always be two or more dependent variables y_j having a common maximum value of τ_j^2/h_j^2. That is there will never be a single variable which is more likely to go out of its control band than at least one of the remaining ones.

The minimax method will therefore result as a rule in an upper bound for P, the probability of satisfactory operation, which is higher than the probability for the other three control schemes, but at the expense of many more computations. It is for this reason that the minimax calculations should be attempted only if none of the other three methods mentioned earlier appear promising.

EXAMPLE: PRODUCTION CONTROL WITH LIMITED STORAGE

To fix the theoretical ideas developed in the preceding section, the authors now present the detailed study of a particular overdetermined system for which no automatic control device has yet been proposed, despite the widespread occurrence of systems of this type in the chemical industry. In the chemical plant shown in Figure 1 equal quantities of raw material 1 and 2 react to form a mixture which is separated into two product streams 3 and 4 that amount respectively to 80 and 20% of the total feed. Every 30 days the plant manager receives a prediction of the average supply and demand rates for the month to follow. The actual average rates may be treated as independent random variables normally distributed about the predictions with standard deviations of 100 units/day. Each day the manager knows what the true average rates have been to date. He wishes to regulate the daily feed rate to his reactor so that at the end of the period none of his four tanks are completely empty. In addition he does not want the feed rate to vary more than 400 units/day from the average value corresponding to the predicted demands. Hence the quantities in the four tanks and the flow rate m are all critical. The initial quantities in the tanks are given in Table 1, and to simplify the problem it is assumed that the tanks are large enough to preclude any danger of their overflowing during the 30 day period.

This problem is a little troublesome to formulate because the quantity in each tank must be obtained by integration of the difference between the inflow and the outflow. Thus as time passes the system becomes increasingly difficult to operate satisfactorily. This theoretical difficulty can however be circumvented easily by restricting attention to the state of the system at the end of the 30 day planning period. Let the variables $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$, $\overline{x}_4$, and $\overline{m}$ be the actual average rates for the 30 days. With this convention then the problem takes the form studied in the second part of this paper, and the control constants can be found in the manner already explained. But since the control function is linear in the uncontrolled variables, the same coefficients are valid for any time interval, or indeed, for the instantaneous flow rates themselves. It should be pointed out that this problem is an extension of a more simple short-range production planning problem previously solved (*2*), although it should be noted that earlier solution cannot in general apply to the present case because the four raw material and product rates cannot be varied independently and because the feed rate variations are restricted.

There are several reasons why systems of this sort lend themselves quite well to compromise control. In the first place they are inherently overdetermined, the several quantities in the tanks all being dependent on the adjustments of the single manipulated production rate. Although overdeterminacy of such systems is sometimes overcome in practice by making the tanks and inventories very large, many companies do not wish to invest the considerable amounts of capital needed to do this. In these circumstances inventories are presently regulated by informal compromise control arrangements enforced by the engineers managing the system. Thus an automatic compromise control system should find application here.

Another reason why production control problems are particularly suited to compromise control is that the important parameters of the system, the inventories and tank capacities, are often quite precisely known. This is important because present compromise control theory does not use feedback to correct for inaccuracy in the parameters. Another fortunate characteristic of production planning problems is the linearity of their equations, which are all based on material balances. A final reason for choosing this example is that the disturbances to manufacturing systems, while often large and certainly random, are usually quite slow in comparison with the system's ability to react to them. Hence dynamic effects may realistically be neglected.

To begin the analysis one must derive the system equations (*1*), which express the accumulations Δy_j ($j = 1, 2, 3, 4$) in each tank after 30 days as functions both of the actual errors $\overline{\Delta x}_i$ ($i = 1, 2, 3, 4$) in the pre-

TABLE 1. INITIAL QUANTITIES IN TANKS

Tank i	Initial quantity h_i in tank i	Number of standard deviations h_i/T_i: $\Delta\overline{m} = 0$	$\Delta\overline{y}_4 = 0$	Weighted least squares	Minimax
1	5,000 units	1.67	1.25	2.0	1.85
2	9,000 units	3.00	1.62	2.3	2.05
3	12,000 units	4.00	1.60	3+	2.02
4	4,000 units	1.33	∞	1.5	1.85

TABLE 2. COMPARISON OF CONTROL SCHEMES

Control scheme		$\Delta m = 0$	$\Delta y_4 = 0$	Weighted least squares	Minimax
b_1		0	0	0.56	1.33
b_2		0	0	0.31	0
b_3		0	0	0.23	0
b_4		0	5	0.70	1.67
PROBABILITIES	P_1	0.952	0.895	0.978	0.968
	P_2	0.999	0.947	0.999	0.980
	P_3	1.000	0.945	1.000	0.978
	P_4	0.909	1.000	0.937	0.968
	$P_5(m)$	1.000	0.788	1.000	0.970
Upper bound		0.909	0.788	0.937	0.968
Lower bound		0.864*	0.631	0.915	0.900

* Exact probability of satisfactory operation.

dicted demands on the system and of the deviation $\Delta\bar{m}$ of the average production rate from the one originally predicted. These 30-day material balances give

$$\Delta y_1 = 30\Delta\bar{x} \qquad - 15\Delta\bar{m} \quad (15a)$$

$$\Delta y_2 = \qquad 30\Delta\bar{x}_2 \qquad - 15\Delta\bar{m} \quad (15b)$$

$$\Delta y_3 = \qquad - 30\Delta\bar{x}_3 \qquad + 24\Delta\bar{m} \quad (15c)$$

$$\Delta y_4 = \qquad - 30\Delta\bar{x}_4 + 6\Delta\bar{m} \quad (15d)$$

For any given set of $\Delta\bar{x}_i$ (which are not under control), the above equations are overdetermined, for although it would be convenient to make all four of the Δy_j vanish, only one variable $\Delta\bar{m}$ can be manipulated.

Next the deviation of the manipulated production rate $\bar{m}$ is expressed as a linear function of the fluctuation in the uncontrolled variables $\bar{x}_j$:

$$\Delta\bar{m} = b_1\Delta\bar{x}_1 + b_2\Delta\bar{x}_2 + b_3\Delta\bar{x}_3 + b_4\Delta\bar{x}_4 \quad (16)$$

The proportionality constants b_j are yet to be determined. By combining Equations (15) and (16) one obtains the expressions equivalent to Equation (5):

$$\Delta y_1 = (30 - 15b_1)\Delta\bar{x}_1 - 15b_2\Delta\bar{x}_2 - 15b_3\Delta\bar{x}_3 - 15b_4\Delta\bar{x}_4 \quad (17a)$$

$$\Delta y_2 = - 15b_1\Delta\bar{x}_1 + (30 - 15b_2)\Delta\bar{x}_2 - 15b_3\Delta\bar{x}_3 - 15b_4\Delta\bar{x}_4 \quad (17b)$$

$$\Delta y_3 = 24b_1\Delta\bar{x}_1 + 24b_2\Delta\bar{x}_2 + (-30 + 24b_3)\Delta\bar{x}_3 + 24b_4\Delta\bar{x}_4 \quad (17c)$$

$$\Delta y_4 = 6b_1\Delta\bar{x}_1 + 6b_2\Delta\bar{x}_2 + 6b_3\Delta\bar{x}_3 + (-30 + 6b_4)\Delta\bar{x}_4 \quad (17d)$$

The variances τ_i^2 of the indirectly controlled amounts in the tanks may now be expressed entirely in terms of the unknown constants b_j with Equation (6):

$$\tau_1^2 = (300)^2[1 - 0.5b_1)^2 + 0.25b_2^2 + 0.25b_3^2 + 0.25b_4^2] \quad (18a)$$

$$\tau_2^2 = (300)^2[0.25b_1^2 + (1 - 0.5b_2)^2 + 0.25b_3^2 + 0.25b_4^2] \quad (18b)$$

$$\tau_3^2 = (300)^2[0.64b_1^2 + 0.64b_2^2 + (1 - 0.8b_3)^2 + 0.64b_4^2] \quad (18c)$$

$$\tau_4^2 = (300)^2[0.04b_1^2 + 0.04b_2^2 + 0.04b_3^2 + (1 - 0.2b_4)^2] \quad (18d)$$

The variance τ_m^2 of the manipulated production rate is of course given by

$$\tau_m^2 = (100)^2[b_1^2 + b_2^2 + b_3^2 + b_4^2] \quad (19)$$

The performances of the two conventional and two compromise control schemes proposed earlier may now be evaluated.

Control of the Manipulated Variable

If the production rate were fixed then of all of the constants b_j would be zero, and the standard deviations τ_i of the amounts in the tanks would all be 3,000 units (τ_m would be zero of course). Table 1 (in the column $\Delta\bar{m} =$ 0) shows the quantities originally in the tanks. They are expressed as multiples of these standard deviations of the estimated quantities in the tanks after 30 days during which the production rate is held constant. Since these quantities are normally distributed, one may use any table of the Gauss distribution (6) to ascertain the probability that any tank will still have some material left in it after 30 days. For example tank 3 contains so much originally (4 standard deviations) that it is extremely improbable that it would run dry. On the other hand the quantity in tank 4 is so low (only 1.33 standard deviations) that its probability of still containing something after 30 days is only 0.909. These probabilities, together with those for satisfactory operation in the other tanks and for the production rate itself, are shown in Table 2 (in the $\Delta\bar{m} = 0$ column).

A lower limit to the probability of satisfactory operation of the entire system is obtained by a consideration of the probability of overall satisfactory operation which would apply if the random variables y_i were uncorrelated. This probability would be the product of the five individual probabilities of satisfactory operation, namely, (0.952) (0.999) (1.000) (0.909) (1.000) = 0.864. For this particular method of control the amounts will in fact be uncorrelated, and so the overall probability of satisfactory operation will be exactly 0.864. In general however the variables are often so highly correlated by the action of the manipulated variable that the true probability is usually nearer the upper bound.

Control of the Amount in Tank 4

The preceding control scheme is not very good because the chances are only about six to one that all of the tanks will still contain material after 30 days. The weakest link is tank 4, which only has 4,000 units in it. Suppose one manipulates the production rate to hold the amount in tank 4 constant ($\Delta y_4 = 0$) to see if this will give better performance. Inspection of Equations (17d) or (18d), or direct use of Equation (10), gives the following control constants: $b_1 = b_2 = b_3 = 0$; $b_4 = 5$. This means that fluctuations in the demands on tanks 1, 2, and 3 are ignored, but any demands on tank 4 are fully corrected for. Equations (18) and (19) give the following standard deviations: $\tau_1 = \tau_2 =$ 8,100; $\tau_3 =$ 12,400; $\tau_4 = 0$; $\tau_5 = 500$. That these standard deviations are much too large for practical control is verified in the $\Delta y_i = 0$ columns of Tables 1 and 2, where one sees that the production rate will be inside its limits only 79% of the time. Since the overall probability of satisfactory operation must be less than this, this second scheme is not even as good as the first.

Weighted Least Squares Control

Now it is necessary to try out the weighted least squares method, the simpler of the two compromise control schemes proposed. The proportionality constants b_i are readily calculated from Equation (14). For example b_1 is given by

$$b_1 = \frac{(30)(15)}{(5{,}000)^2}\left[\left(\frac{15}{5{,}000}\right)^2 + \left(\frac{15}{9{,}000}\right)^2 + \left(\frac{24}{12{,}000}\right)^2 + \left(\frac{6}{4{,}000}\right)^2\right]$$

$= 0.56$. The other coefficients are given in Table 2. They are all rather small, with slightly stronger emphasis on protecting tanks 1 and 4. That this protection is not strong enough is shown clearly in Tables 1 and 2, for tanks 1 and 4 are still the most likely to run dry. However the probability of satisfactory operation for the worst tank (tank 4) is now up to 0.937, and since the dependent variables will tend to be highly correlated by the manipulations of the production rate one would expect the probability of satisfactory overall operation to approach this level. The lower bound on this probability, which would be obtained if all levels varied independently, is 0.915, higher than the upper bounds for any of the conventional schemes. There is still about one chance in twelve that a tank will be dry after 30 days.

Minimax Control

Since there is still room for improvement in this system, the rather involved calculation of the minimax constants is justifiable. The constants were calculated by Maurice Vargaftig, using a IBM-650 computer program which will handle systems having up to four uncontrolled variables (5). This program simply mechanizes the algorithm described elsewhere (2). In this case minimax control acts only to control tanks 1 and 4 and ignores secondary effects on the other three critical variables. It will in fact cause the levels in these two tanks to move up and down together in perfect correlation, neither level dropping to zero before the other. The levels in the other tanks will, on the other hand, vary independently from each other and from the other variables. A lower limit to the overall probability of satisfactory operation will be therefore (0.968) (0.980) (0.978) (0.970) = 0.900.

As expected, minimax control raises the probability of satisfactory operation in the most sensitive tank from 0.937 to 0.968. The upper bound for minimax control is therefore higher than that for any of the other schemes, including even the other compromise method involving weighted least squares. On the other hand the lower bound for minimax control is not quite as good as the one for weighted least squares. A simulation of the two control schemes on a digital computer did not show any significant difference between minimax and weighted least squares control that would justify the more extensive calculations needed to obtain the minimax control constants. In this case both compromise techniques give performance which is apparently quite close to the best attainable. Reference 3 describes a situation where minimax control exhibits a clear advantage over weighted least squares control.

It is interesting to consider other equivalent measures of performance in order to understand more fully the advantage of minimax control over the more simple method of holding the feed rate constant, a practice widely used today. Thus with a constant feed rate there is one chance in seven of having an emergency situation after 30 days but only about one chance in fifteen when the minimax control is chosen. On the other hand one may compare the schemes on the basis of the maximum length of undisturbed planning period which is attainable for a fixed probability level. After only 21 days the constant feed control probability drops below that maintained after 30 days by minimax control. Or, if the 30 day performance of the conventional scheme is acceptable, one could extend the planning period to 42 days with minimax control and have an equivalent reliability.

Finally one must consider how much of an increase in the initial amounts would be needed to bring the performance of the conventional scheme up to that of the minimax method. The initial quantity in tank 1 would have to be increased 23% (from 5,000 to 6,170 units) and that in tank 4, 38% (from 4,000 to 5,550 units).

There is a final point of theoretical interest. In this case the minimax scheme gives the same control function as the simpler methods of reference 2, in which restrictions on the feed rate were not considered at all. This however is fortuitous, for it would have been difficult to predict a priori that this simplification would have been possible. Thus a similar system has been treated in reference 3, in which this coincidence does not occur, and where there are three tanks simultaneously limiting the system performance. As a matter of fact a rather artificial example has also been constructed that forces the minimax control to consider all four tank levels (5).

CONCLUSIONS AND RECOMMENDATIONS

A theory has been developed for describing and evaluating linear overdetermined control systems subject to slow Gaussian perturbations. In the case presented the proposed compromise control schemes demonstrated superior performance relative to conventional methods, at least as measured by the probability of satisfactory operation.

In general it would only seem wise to use the compromise methods, which require some kind of computation, if the conventional control schemes do not result in an acceptable probability of satisfactory operation. But one can certainly imagine situations where the cost of the more complicated compromise control system would be justified by the savings in capital expenditures made possible by controlling several variables simultaneously with only one manipulated parameter. And in production scheduling systems compromise control can significantly lengthen the time a plant is able to operate without running into an emergency situation.

NOTATION

a_{ij} = sensitivity of j^{th} interior variable to i^{th} exterior variable
b_i = control constant for i^{th} exterior variable
c_j = sensitivity of j^{th} interior variable to manipulated variable
h_j = nearer limit on j^{th} interior variable
h'_j = farther limit on j^{th} interior variable
m = manipulated variable
P_j = probability of satisfactory operation for j^{th} variable
P = probability of satisfactory operation for system
σ_i = standard deviation of i^{th} exterior variable
τ_j = standard deviation of j^{th} interior variable

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